**Barron’s Let’s Review Regents – Algebra II**

# Chapter 4: Radical Expressions and Equations

## 4.1 Simplifying Radicals

**Key Ideas**

A *radical expression* is one that involves a sign (radical sign). Radical expressions are often involved in the solutions to polynomial equations. To work with radicals, you must know how to put them into simplified form and how to combine them.

**Definition of Radicals**

The square root of a number is the thing that must be multiplied by itself to get that number. The symbol for square root is , also called *the radical sign*. An example is because . Even though it is also true that , the symbol means just the positive number that when squared is equal to the number inside the radical sign. A number that is not a perfect square, like 7, still has a square root, although that square root is an irrational number. is between 2 and 3 since and . More precisely, is approximately 2.645751311.

If there is a small number outside the radical sign, it no longer indicates a square root. If there is a small 3 outside the radical sign, it becomes a cube root sign and is equal to the number that must be cubed to become the number under the radical sign. An example is , because . The small number outside the radical sign is called the *index*. When there is no index, it is implied to be a 2. So a radical sign with no index is called the *square root sign*.

**Multiplying Radicals**

Two radical expressions that have the same index can be multiplied by multiplying the numbers inside the radical sign. For example, . This is easily verified for this example since .

**Math Facts**

In general, . This rule also works in reverse for factoring radicals: .

**Simplifying Square Roots**

If the number inside a square root sign has a factor that is a perfect square, the radical can be *simplified*. The can be simplified since one of the factors of 50 is 25, which is a perfect square.

. The 5 in this expression is not an index but a coefficient in front of the radical sign. The multiplication sign between the coefficient and the radical sign is not necessary.

**Example 1**

Simplify the expression .

*Solution*: Since .

It is not always clear whether or not a large number has a factor that is a perfect square. By factoring the number into its *prime factors*, it is possible to group the matching factors into pairs and use the fact that .

Radical expressions that involve variables can also be simplified with this approach.

**Example 2**

Simplify the expression:

*Solution*:

**Adding and Subtracting Radicals**

Radicals can be added or subtracted only if they have the same index and the same number inside the radical sign. They are combined the same way that like terms are combined with polynomials. For example, . Subtraction works the same way: .

If two radical expressions have the same index but different numbers inside the radical sign, you cannot immediately add or subtract them. Sometimes after simplifying the expressions, they will have the same number inside the radical sign and can then be added or subtracted.

**Example 3**

Simplify the terms and combine if possible.

*Solution*: After simplifying:

### Check Your Understanding of Section 4.1

1. Multiple-Choice
2. What is ?  
   **(2) 71**
3. Which of the following is equivalent to ?  
   **(4)**
4. Which of the following is equivalent to ?  
   **(1)**
5. What is ?  
   **(2)**
6. What is ?  
   **(1)**
7. Which of the following is equivalent to ?  
   **(4)**
8. Which of the following is equivalent to   
   **(2)**
9. What is ?  
   **(3) 71**
10. Simplify .  
    **(2)**
11. What is ?  
    **(1) 30**
12. *Show how you arrived at your answers*.
13. Spencer simplifies as   
    . Mia simplifies it as .  
      
    **Mia is correct. There is no distributive property for radicals.  
      
    Radicals can be added or subtracted only if they have the same index and the same number inside the radical sign.  
      
    The square root of a sum is not equal to the sum of the square roots: .**
14. Daniel calculates . Is this correct? Why or why not?  
      
    **No. The square root of a sum is not equal to the sum of the square roots:   
    .  
      
    Radicals can be added or subtracted only if they have the same index and the same number inside the radical sign.**
15. In this right triangle, what is the length of leg simplified in simplest terms?
16. What is
17. What is ?

## 4.2 Imaginary and Complex Numbers

**Key Ideas**

An *imaginary number* is a number that, when multiplied by itself results in a negative number. The most basic imaginary number is since . The number is abbreviated by the symbol *i*. Based on this definition, . When imaginary numbers are combined with real numbers, the result is called a *complex number*. Complex numbers can be added, subtracted, and multiplied with rules similar to polynomial adding, subtracting and multiplying.

**Simplifying Imaginary Numbers**

The square root of any negative number can be represented as an expression involving an . To simplify , split it into .

If the number inside the radical is not a perfect square, like . If the number inside the radical has a factor that is a perfect square, like , the will be written between the coefficient and the radical.  
.

**Adding and Subtracting Imaginary Numbers**

To add or subtract to imaginary numbers like , they are considered to be like terms. So they can be combined as you would do with variables,   
. For subtraction, it would be   
.

**Multiplying Imaginary Numbers**

Multiplying imaginary numbers like is similar to multiplying expressions like . Since would be , is . Since ,   
.

**Example 1**

Multiply .

*Solution*: .

**Powers of**

can be raised to other powers besides 2. When is raised to an integer power, the only possible values are or 1.

|  |  |  |
| --- | --- | --- |
| **Power** |  | **Solution** |
| 0 |  | 1 |
| 1 |  |  |
| 2 |  | -1 |
| 3 |  | -i |
| 4 |  | 1 |

After the power of 3, the powers of cycle between the for values: and . raised to any multiple of 4 will be 1.

To raise to a high power, like , find the multiple of 4 smaller than 47 that is closest to it and rewrite as  
.

**Complex Numbers**

When a real number, like 3, is added to an imaginary number, like , together the sum is written as . A complex number is a number of the form , where and are real numbers.

**Adding and Subtracting Complex Numbers**

Complex numbers are added by adding the real parts and the imaginary parts separately and writing the answer in form.

Subtracting complex numbers requires care as you distribute the -1 through the second expression.

**Multiplying Complex Numbers**

Complex numbers can be multiplied the way binomials are multiplied by using the FOIL shortcut. Anytime an is encountered in a solution, it should be converted to a -1.

To multiply with FOIL:

**Complex Solutions to Quadratic Equations**

If in the process of using the quadratic formula a negative number appears inside the radical, the quadratic equation is said to have two complex solutions.

**Example 4**

Use the quadratic equation to solve   
 with and .

Solution:

Since there is a negative number inside the radical sign, the solutions will be complex numbers.

There are two complex solutions:   
.

**Graphing Complex Numbers on the Complex Plane**

A number like cannot be graphed on the standard number line. Instead, complex numbers are graphed on something called *the complex plane*. The complex plain is like a number line for complex numbers. The complex plane resembles the two-axis coordinate plane. Instead of the axes representing the x-coordinate and the y-coordinate, the axes represent the real part of the complex number and the imaginary part of the complex number.

Below is the complex plane. The number is plotted as a single point in the position (2, 3).

Real Numbers like 2 can be thought of as and are graphed on the real axis of the complex plane. Imaginary numbers like can be thought of as and are graphed on the imaginary axes of the complex plane.

**Math Facts**

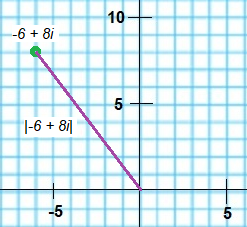
The absolute value of a complex number is the distance on the complex plane between that number and the number . The absolute value of a complex number can be calculated with the formula .

**Example 5**

What is ?  
(1) 5 (2) -5 (3) 7 (4) -7

*Solution*:   
So choice (1) is the answer.

### Check Your Understanding of Section 4.2

1. Multiple-Choice
2. What is ?  
   **(1)**
3. What is ?  
   **(4)**
4. What is ?  
   **(1) 21**
5. What is ?  
   **(1) -1**
6. What is ?  
   **(4)**
7. What is ?  
   **(2)**
8. What is ?  
   **(2)**
9. What is ?  
      
   **(1) 34**
10. Solve the quadratic equation   
    .  
    **(1)**
11. What is ?  
    **(3) 13**
12. *Show how you arrived at your answers*.
13. Plot the point on the complex plane. What is the absolute value of ?  
      
    
14. Find the 4 answers to the equation   
    .
15. What is
16. What is ?
17. A complex number whose absolute value is greater than 2 is not in something called the Mandelbrot Set. Show that is not in the Mandelbrot Set.

## 4.3 Solving Radical Equations

**Key Ideas**

A radical equation is one that involves the square root, or cube root, or other kind of root of a variable. If a radical equation also involves the same variable raised to the first power, the equation has multiple solutions One step in a radical equation is usually to, at some point, square, or cube, both sides of the equation.

**One-Step Radical Equations**

When you square the square root of a perfect square, the result is the number that was originally under the radical sign. So . This is also true for numbers that are not perfect squares like . In general, we can say

The equation can be solved by squaring the expression on both sides of the equal sign.

**Two-Radical Equations**

Before squaring both sides of a radical equation, the radical term must be isolated. In the equation   
, the must be isolated by subtracting 3 from both sides of the equation. Then the equation an be completed by squaring both sides of the equation to eliminate the radical sign.

If the expression under the radical sign is not simply an , there will be additional steps after the radical sign has been eliminated.

**Radical Equations Involving a Linear Term**

A more complicated type of radical equation is one like . By testing different values, it can be seen that is the number that makes the equation true since . A different approach is needed if the equation requires an algebraic solution, if there is more than one solution, or if the solutions are irrational numbers.

As with the simpler equations, the first step is still to isolate the radical term. For this example, this means to subtract from both sides of the equation.

Just like before, the next step is to eliminate the radical sign by squaring both sides of the equation. Squaring the right side, however, involves using polynomial multiplication (FOIL) described in Chapter 1.

The process did get the solution , but it also seems to have found another solution. When you substitute into the original equation, the solution does not work out properly.

The solution is not true!

When this happens, the number is not an answer to the equation and we not include it in the solution set. The clearest way to indicate this on a test is to cross out the solution and write the word reject.

The only solution to this equation is .

**Math Facts**

Squaring both sides of an equation can sometimes make equations that are not true into equations that are true. For example, it is not true that , but it is true that . This is why it is necessary to check any solutions that come from solving a radical equation that eventually became quadratic equation.

**Example 4**

What is the solution set for ?  
(1) {2, 8}  
(2) {8}  
(3) {2}  
(4) {}

Solution: Choice (3) is the correct answer.

This question an be solved with algebra like the others and will eventually become a quadratic equation with two solutions . However, the solution will need to be rejected since it does not make the original equation true.

Since this is a multiple-choice question, the quickest and most accurate way to answer it would be to check the two numbers to see that only makes the original question true.

**Radical Equations involving Two Radical Terms**

Solving an equation that involves two radical terms like is a lengthy process. It is tempting to try to square both sides to eliminate all the radicals. However, this doe not work since after squaring the left side with FOIL, there will outer and inner terms to deal with.

To solve this sort of equation, first isolate one of the radical terms. Then square both sides very carefully. Continue by isolating the other radical term and squaring both sides again.

Check to see if the answer should be rejected.

The solution is .

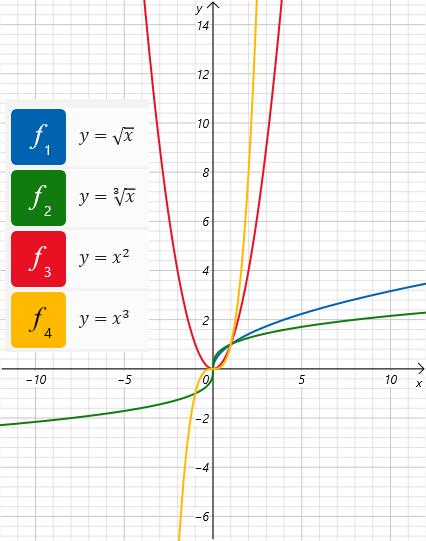
### Check Your Understanding of Section 4.3

1. Multiple-Choice
2. What is the solution to ?  
   **(2) 49**
3. What is the solution to ?  
   **(3) 79**
4. What is the solution to ?  
   **(2) 18**
5. What values of make the equation   
    true?  
   **(3) 16**
6. What value(s) of make the equation   
    true?  
   **(2) 25**
7. Find all solutions to the equation   
   .  
   **(1) 5, -1**
8. Find all solutions to the equation   
   .  
   **(3) 6**
9. Find all solutions to the equation  
   .  
   **(1)**
10. For what value(s) of ?  
    **(4) 1, 0**
11. Solve for : .  
    **(2) 7**
12. *Show how you arrived at your answers*.
13. Solve for .
14. Solve for .  
    **Solution:**
15. The equation has a solution set of {3}. If you square both sides of the equation, what is the solution set of the new equation?  
    **Solution set of new equation: {-3, 3 }**
16. Jace tried to solve the equation  
     by first squaring both sides. He got:   
      
    However, 8 is not a solution to the equation. What did Jace do wrong?  
      
    Jace did not square the radical expression properly, but more importantly, he did not follow proper procedure by isolating the radical to one side of the equation.
17. What value(s) of satisfy this equation?

## 4.4 Graphs of Radical Functions

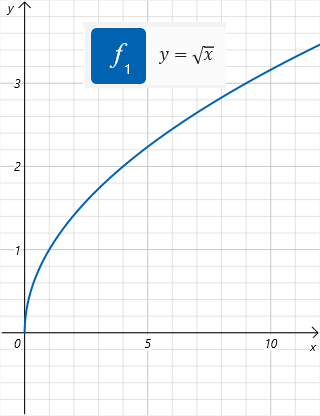
**Key Ideas**

The graphs of equations involving a radical like   
 or have shapes that are related to the graphs of the polynomial graphs and , respectively. Graphs of functions involving radicals can be transformed by shifting left, right, up, or down and through vertical stretches and horizontal squeezes.

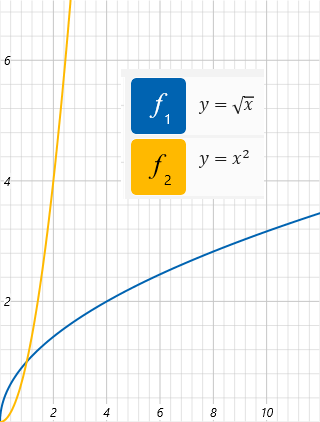


**The Graph of the Square Root Function**

The graph of the function includes the points (0, 0), (1, 1), (4, 2) and (9, 3). The symbol evaluates only to the positive square root of a number, the -coordinates of all the points (the *range*) are all numbers greater than or equal to 0. Though the concept of imaginary numbers was introduced in the last section for the purpose of graphing on the coordinate axes, there is no place to graph a point like (-4, 2). So the -coordinates of the points on this graph (the *domain*) will also be all numbers greater than or equal to 0. For this reason, the entire graph will be in quadrant I.

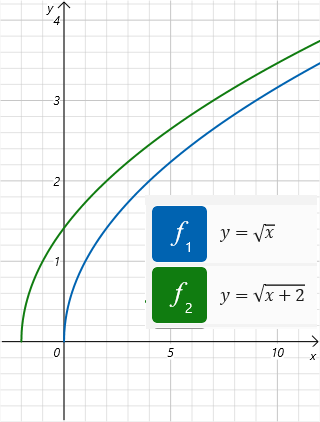


This curve is actually a half of a parabola reflected over the line . Unlike the graphs of the polynomial functions introduced in Chapter 1, this graph’s domain is not all real numbers but is .



**Transformations of the Square Root Function**

The function has a graph very similar to the graph of . In fact, can be written as . The graph of is the result of the shifting the graph of to the *left* by 2 units.



The four most common transformations of the function are shown in the following table.

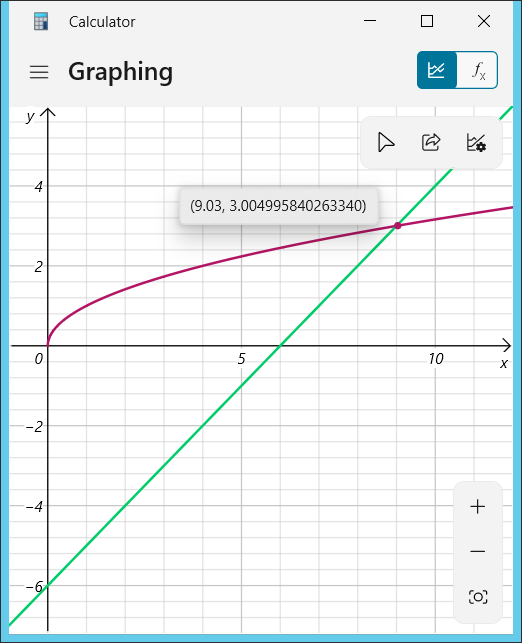
|  |  |
| --- | --- |
| shifted to the *left* 2 units. | *vertical stretch* by a factor of 2 |
| shifted to the *right* 2 units | horizontal *squeeze* by a factor of 2 |

**Graphically Solving Radical Equations**

In Section 4.3, radical equations like were solved algebraically. Radical equations can also be solved with the intersect feature of the graphing calculator.

To get the most useful graph to analyze, first isolate the radical term by adding to both sides.

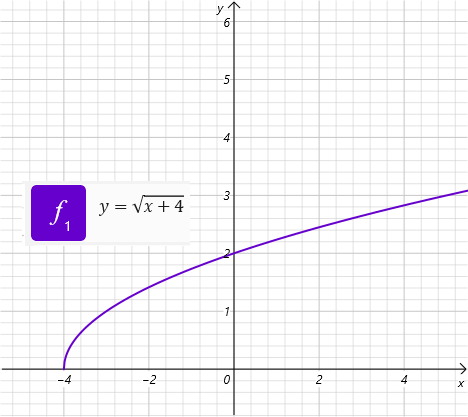
**Solution:**

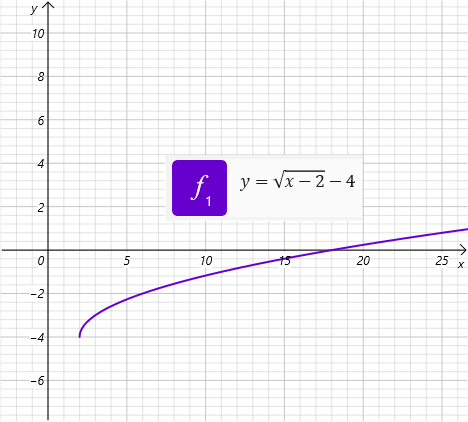


### Check Your Understanding of Section 4.4

1. Multiple-Choice
2. Which graph is a sketch of ?  
   (0, 2)  
   **(3)**
3. Which graph is a sketch of ?  
   Points: (-2, 0), (2, 2)  
   **(2)**
4. Which graph is a sketch of ?  
   Points: (0, -2), (4, 0)  
   **(1)**
5. Which graph is a sketch of ?  
   Points: (2, 0), (6, 2)  
   **(4)**
6. Which equation produces the following graph?  
   Points: (-3, -1), (1, 1)  
   **(2)**
7. Which equation produces the following graph?  
   Points: (3, 1), (4, 2)  
   **(1)**
8. Solve the equation by graphing:   
      
   **(1) 6**
9. Solve the equation by graphing:   
   **(1) (5, -1)**
10. Solve the equation by graphing.  
    **(3) 7**
11. Which of the following is a point on the graph of ?  
    **(1) (8, 17)**
12. Show how you arrived at your answers.
13. Sketch the graph of on the axes below.

|  |  |
| --- | --- |
| x | y |
| -4 | 0 |
| 0 | 2 |
| 5 | 3 |
| 12 | 4 |
| 21 | 5 |



1. What is the equation for this graph?  
   Points: (2, -4),(3, -3), (6, -2)  
   (h, k) = (2, -4)  
   [Writing Equation Given a Square Root Graph](https://www.youtube.com/watch?v=tfnOzSf_E_k)  
     
   
2. Ariana thinks that this graph is a vertical stretch by a factor of 2 of . Sienna thinks that it is a horizontal squeeze by a favor of 4. Who is correct?  
   Points: (0, 0), (4, 4), (9, 6)

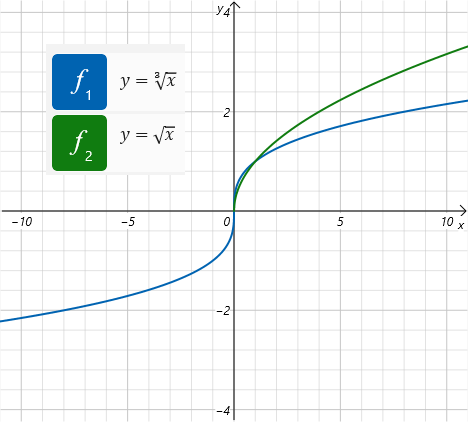
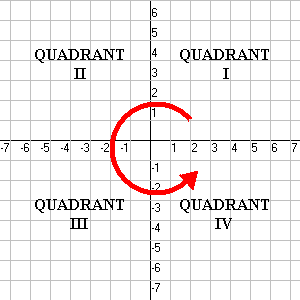
|  |  |
| --- | --- |
| x |  |
| 0 | 0 |
| 4 | 4 |
| 9 | 6 |

Vertical Stretch matches the points.  
Equation for Vertical Stretch matches the equation given   
  
Horizontal Squeeze:

|  |  |
| --- | --- |
| x |  |
| 0 | 2 |
| 4 | 4 |
| 9 | 5 |

Horizontal transformations shift the solution either right or left, but vertical solutions do not.

The horizontal squeeze equation does not fit the points on the graph. No addition or subtraction to will have the first point being (0, 0).

1. How does the graph of compare to the graph of .  
     
     
     
     
    on has points in the Quadrant I (positive x and positive y values).  
     
    has points in Quadrant I and III (positive x and y values or negative x and y values).
2. Based on this graph, what equation has a solution of ?  
     
   Points: (-2, 3), (-1, 4), (7, 6)  
   The point (-1, 4) increases the x value by 1 unit and the y unit by 1, indicating that

|  |  |
| --- | --- |
| x |  |
| -2 | 3 |
| -1 | 4 |
| 0 |  |
| 7 | 6 |

